Optimal Design of a Vibrating Beam with Coupled Bending and Torsion

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The problem of maximizing the fundamental frequency of a thin-walled beam with coupled bending and torsional modes is studied in this paper. An optimality criterion approach is used to locate stationary values of an appropriate objective function subject to constraints. Optimal designs with and without coupling are discussed.

Nomenclature

b	= width of a cross section
h	= height of a cross section
n	= positive exponent
q_i	= ith degree of freedom for the system in coupled bending-torsion vibration
t_f	= flange thickness
t_w	= web thickness
t_r^e	= rth design variable in element e
v_r, w_r	= displacements of the reference axis
v_s, w_s	= displacement of the shear center
$\{\bar{w}_r\}, \{\bar{w}_s\}$	= nodal 4×1 displacement vectors
$\{w_r\}, \{w_s\}$	= elemental 8×1 displacement vectors
A	= cross-sectional area
C_{wr}	= warping constant
E	= modulus of elasticity
G	= shear modulus
I_{pr}	= polar moment of inertia about the web center
I_y, I_z	= principal moments of inertia about the reference axis
I_{zw}	= first moment of the sectorial area about the y axis
J	= torsion constant
K_{ij}, M_{ij}	= element in the ith row and jth column of the global stiffness and mass matrices, respectively
K_v, M_v	= global stiffness and mass matrices, respectively; developed with nodal degrees of freedom v_s and dv_s/dx
K_w, M_w	= global stiffness and mass matrices, respectively; developed with nodal degrees of freedom w_r, dw_r/dx, θ, $d\theta/dx$
K_{wr}^e, M_{wr}^e	= elemental stiffness and mass matrices, respectively; developed with nodal degrees of freedom w_r, dw_r/dx, θ, $d\theta/dx$
$\bar{K}_{ws}^e, \bar{M}_{ws}^e$	= elemental stiffness and mass matrices, respectively; developed with nodal degrees of freedom w_s, dw_s/dx, θ, $d\theta/dx$
L	= total length of the beam
L^e	= elemental length of the beam
$\bar{M}$	= specified mass constraint
M_y, M_z	= bending moments
M_{wr}	= bimoment due to restrained warping
M_{xsv}	= St. Venant twisting moment
N	= total number of elements
N_q, N_w, N_v	= total number of equilibrium equations
N_t, N_r	= total number of optimality criteria
$N\phi$	= total number of design variables
N_{cr}	= total number of elements with specified cross-section area
S_z	= first moment of the area about the z axis
T_{max}	= maximum total kinetic energy
$\bar{T}_{max}$	= reference kinetic energy
$T, \bar{T}$	= transformation matrices
U_{max}	= maximum kinetic energy
$\bar{V}$	= specified volume constraint
V_c	= total volume of elements with constrained area
ϵ	= prescribed tolerance
η	= positive exponent
ν	= iteration count
θ	= angle of twist
ρ	= density
$\alpha, \beta, \lambda, \Omega$	= Lagrange multipliers
$\bar{\lambda}$	= estimated value of λ
ϕ_j	= jth design variable
$\phi_j\|_{min}$	= specified lower bound on the design variable
$\phi_j\|_{max}$	= specified upper bound on the design variable
ω	= natural frequency of vibration
ω_v	= frequency of the system in uncoupled vibration with bending in one direction
ω_w	= frequency of the system in coupled bending-torsion vibration

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I. Introduction

A FIRST investigation of the optimal beam vibration problem is attributed to Niordson.[1] He considered the problem of finding the best taper that yields the highest possible natural frequency. Following the initial work of Niordson, many different investigators have considered different problems in the field of optimal vibrations of beams.[2-20] References 2–8 are concerned with the maximization of fundamental frequencies. Olhoff[9,10] has addressed the problem of maximizing higher-order frequencies and rotating beams.[11] The problem of minimizing weight for a specified frequency constraint has been addressed in Refs. 12–18. Multiple frequency constraints have been addressed in Refs. 19–23. An optimality criteria approach has been discussed in Refs. 17 and 18.

An application to the helicopter blade design problem has been presented by Peters et al.[24] In their work, the problem of optimum distribution of mass and stiffness for a frequency constraint has been discussed. In most cases this is the dual of the problem of maximizing the frequencies,[12-16] which is considered as a primal problem. It is possible to solve several primal problems to obtain a solution to a dual problem. Either of these approaches results in an optimum design and a structural dynamic model corresponding to the optimal design.

The resulting mathematical model can be used as a model for tests and improvements of these models by identification techniques. In an application of this[24] and in all other optimal vibration problems, only uncoupled vibration modes have been considered. In the helicopter design problem[24] and many other practical situations, elastic axes do not coincide with the inertial axes, resulting in a coupling between some of the bending modes and torsional modes. This paper addresses the problem of maximizing the fundamental frequency of a thin-walled beam with coupled bending and torsional modes. This is achieved through an optimality criterion approach to locate stationary values of a proper objective function. The results show that the optimum designs are very different from the design obtained for beams with uncoupled vibration, showing that the coupling must not be ignored in the optimization process.

II. Primal Optimization Problem for a Continuous System

A beam of channel cross section with one axis of section symmetry is considered (Fig. 1). Reference 25 presents a formulation suitable for beams in which flange and web thicknesses and material properties are nonuniform in a cross section. However, in this paper the thicknesses and properties are specified to be uniform in a cross section. If such a beam is vibrating in simple harmonic motion with frequency ω, the maximum strain energy and the maximum kinetic energy (see Appendix) are

$$2U_{\max} = \int_L \left[EI_y \left(\frac{d^2 w_r}{dx^2} \right)^2 + 2EI_{zw} \frac{d^2 w_r}{dx^2} \frac{d^2\theta}{dx^2} + EC_{wr} \left(\frac{d^2\theta}{dx^2} \right)^2 + GJ \left(\frac{d\theta}{dx} \right)^2 + EI_z \left(\frac{d^2 v_r}{dx^2} \right)^2 \right] dx \tag{1}$$

$$2T_{\max} = \omega^2 (2\bar{T}_{\max}) \tag{2}$$

with

$$2\bar{T}_{\max} = \int_L \rho (A w_r^2 + 2S_z w_r \theta + I_{pr}\theta^2 + A v_r^2) dx \tag{3}$$

Nonstructural concentrated masses are included in Ref. 25. From the requirement that $2U_{\max} = 2T_{\max}$, and with the constraint that $2\bar{T}_{\max} = 1$, it follows that

$$\omega^2 = 2U_{\max} \tag{4}$$

For the optimization process, $\phi_j(x)$, $j = 1,2,...,N$, denotes the jth design variable, limited in this paper to the flange and web thicknesses. The primal problem is to determine the wall thicknesses that provide the maximum value of the fundamental frequency subject to the constraint that the beam mass be equal to some specified value. The formulation of equations is as follows. Maximize

$$\omega^2 = \int_L \left[EI_y \left(\frac{d^2 w_r}{dx^2} \right)^2 + 2EI_{zw} \frac{d^2 w_r}{dx^2} \frac{d^2\theta}{dx^2} + EC_{wr} \left(\frac{d^2\theta}{dx^2} \right)^2 + GJ \left(\frac{d\theta}{dx} \right)^2 + EI_z \left(\frac{d^2 v_r}{dx^2} \right)^2 \right] dx \tag{5}$$

subject to the constraints of satisfaction of equilibrium equations

$$\frac{d^2}{dx^2}\left[EI_y \frac{d^2 w_r}{dx^2} + EI_{zw} \frac{d^2\theta}{dx^2} \right] - \omega^2 \rho (A w_r + S_z \theta) = 0$$

$$\frac{d^2}{dx^2}\left[EI_{zw} \frac{d^2 w_r}{dx^2} + EC_{wr} \frac{d^2\theta}{dx^2} \right] - \frac{d}{dx}\left(GJ \frac{d\theta}{dx} \right) - \omega^2 \rho (S_z w_r + I_{pr}\theta) = 0$$

$$\frac{d}{dx^2}\left[EI_z \frac{d^2 v_r}{dx^2} \right] - \omega^2 \rho A v_r = 0 \tag{6}$$

with appropriate boundary conditions. There is a normalization constraint

$$\int_L \rho (A w_r^2 + 2S_z w_r \theta + I_{pr}\theta^2 + A v_r^2) dx - 1 = 0 \tag{7}$$

The beam mass is specified as

$$\int_L \rho A \, dx - \bar{M} = 0 \tag{8}$$

and there are possible limits on magnitudes of design variables

$$\phi_j |_{\min} \le \phi_j \le \phi_j |_{\max} \tag{9}$$

This problem will be solved with the optimality criterion approach, with the criterion developed by applying the techniques of calculus of variations and Lagrange multipliers, as follows. A modified frequency function F is defined as follows:

$$F[w_r, \theta, v_r; \phi_j] = \omega^2 - \Omega(2\bar{T}_{\max} - 1) - \lambda\left(\int_L \rho A \, dx - \bar{M} \right) \tag{10}$$

That is, the normalization and constant mass constraints are incorporated with Lagrange multipliers Ω and λ, respectively. The problem now is to determine those functions w_r, θ, v_r, and ϕ_j that give a stationary value to the functional F, subject to equilibrium constraints.

First, the variations of the displacements w_r, θ, and v_r are considered. A typical first variation of F will be

$$\tfrac{1}{2}\delta F_{w_r} = \int_L \left[EI_y \frac{d^2 w_r}{dx^2} + EI_{zw} \frac{d^2\theta}{dx^2} \right] \frac{d^2 \delta w_r}{dx^2} dx - \Omega \int_L \rho (A w_r + S_z \theta) \delta w_r \, dx \tag{11}$$

After integration by parts and inclusion of the equilibrium equation constraints, it can be shown that $\delta F_{w_r} = 0$ for every

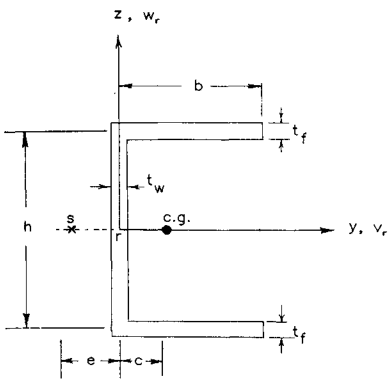

Fig. 1 Section geometry.

δw_r only if $\Omega = \omega^2$. This same requirement follows from $\delta F_\theta = 0$ and $\delta F_{v_r} = 0$.

Finally, variations of the design variables ϕ_j are considered. It is to be noted that variations of a particular design variable are taken only in those regions of the beam domain in which that variable does not have a limiting value set by Eq. (9):

$$\delta F_{\phi_j} = \delta\omega^2_{\phi_j} - \omega^2 \delta(2\bar{T}_{\max})_{\phi_j} - \lambda \int_L \rho \frac{\partial A}{\partial \phi_j} \delta\phi_j \, dx \tag{12}$$

After evaluating the variations, the requirement that $\delta F_{\phi_j} = 0$ for every $\delta\phi_j$ leads to the optimality criterion for each design variable:

$$H_j[w_r(x), \theta(x), v_r(x), \phi_k(x)] = \lambda, \qquad j = 1,2,\ldots,N \tag{13}$$

with

$$\begin{aligned} H_j = \left(\rho \frac{\partial A}{\partial \phi_j}\right)^{-1} \Bigg[& \left(\frac{d^2 w_r}{dx^2}\right)^2 \frac{\partial (EI_y)}{\partial \phi_j} + \left(\frac{d^2 v_r}{dx^2}\right)^2 \frac{\partial (EI_z)}{\partial \phi_j} \\ & + \left(\frac{d^2\theta}{dx^2}\right)^2 \frac{\partial (EC_{wr})}{\partial \phi_j} + \left(\frac{d\theta}{dx}\right)^2 \frac{\partial (GJ)}{\partial \phi_j} \\ & + 2 \frac{d^2 w_r}{dx^2} \frac{d^2\theta}{dx^2} \frac{\partial (EI_{z\omega})}{\partial \phi_j} - \omega^2 \rho \Bigg(w_r^2 \frac{\partial A}{\partial \phi_j} \\ & + 2 w_r \theta \frac{\partial S_z}{\partial \phi_j} + \theta^2 \frac{\partial I_{pr}}{\partial \phi_j} + v_r^2 \frac{\partial A}{\partial \phi_j} \Bigg) \Bigg] \end{aligned} \tag{14}$$

In other words, the optimum design is supposedly achieved when the quantity H_j is constant along the span of the beam for all regions in which the associated ϕ_j does not have a limiting value.

The formulation is summarized as follows. The unknowns are three displacement functions (w_r, θ, v_r), N_ϕ design variable functions ϕ_j, the frequency of vibration ω^2, and the Lagrange multiplier λ. Available equations are three equilibrium equations with associated boundary conditions [Eq. (6)], N_ϕ optimality criterion equations [Eqs. (13) and (14)] or the limiting values [Eq. (9)], the normality condition [Eq. (7)], and the constant mass constraint equation [Eq. (8)]. The problem seems to be well posed, and a simultaneous solution of all equations will lead to possible optimum designs.

Equation (6) shows the decoupling between displacement v_r and the displacement pair w_r, θ. There are two separate eigenvalue problems, leading to eigenvalue ω_v^2 with eigenvector $\hat{v}_r$, with $w_r = 0$, $\theta = 0$, and eigenvalue ω_w^2 with eigenvector $\hat{w}_r$ and $\hat{\theta}$ with $v_r = 0$. If $\omega_v^2 = \omega_w^2$, then the eigenvector will contain nonzero components for all displacements, $\hat{w}_r$, θ, and $\hat{v}_r$.

Now, if the physics of the problem is such that one need optimize only for vibration in the plane of symmetry, then it is permissible to set $w_r = 0$, $\theta = 0$. Such single-displacement optimization problems have been treated many times in the past, most often with cross-section area as the design variable. Equations (13) and (14) will provide the proper optimality criteria for other design variables, such as wall thickness.

Likewise, if it is necessary to optimize only for the coupled vibration, then one may set $v_r = 0$ in Eqs. (13) and (14) to obtain the correct optimality citerion. This coupled-displacement optimization has not been done before, and the reported numerical results in this paper are limited to this problem. The decoupling between v_r and w_r, θ will lead to valid optimum designs in the following two situations.

If the optimality criteria are satisfied with $v_r \neq 0$, $w_r = 0$, $\theta = 0$, and if the optimized ω_v^2 is less than the bending-torsion frequency ω_w^2, then the design is truly optimum. The lowest natural frequency has been raised to the highest value possible.

If Eqs. (13) and (14) are satisfied with $w_r \neq 0$, $\theta \neq 0$, $v_r = 0$, and if the optimized $\omega_w^2 < \omega_v^2$, the design is truly optimum. The lowest frequency has been raised.

However, if Eqs. (13) and (14) are satisfied with $v_r \neq 0$, $w_r = 0$, $\theta = 0$, and the optimized $\omega_v^2 > \omega_w^2$, or if $w_r \neq 0$, $\theta \neq 0$, $v_r = 0$, and the optimized $\omega_w^2 > \omega_v^2$, then the designs are not valid. In either case, the design is such that the optimized frequency is not the fundamental frequency, which means that the fundamental frequency has not been optimized.

If decoupled optimization does not provide the optimum design, then the problem must be reformulated. This observation can be explained by beginning an optimization problem with a cross section of specified depth h, width b, mass $\bar{M}$, and uniform wall thickness t such that $\omega_w^2 < \omega_v^2$. In this case, optimization will attempt to raise ω_w^2 by varying the wall thicknesses. This search for the best wall thicknesses can be thought of as a movement through a design space of thicknesses, seeking the point which provides the largest ω_w^2.

Table 1 Optimum frequency for simply supported beam

No. of elements	ω_1, rad/s	Increase[a,b] in ω_1, %
10	222.1	46.31
20	222.0	46.24
30	221.9	46.18
40	221.7	46.04
50	221.6	45.98
60	221.5	45.92
70	221.4	45.85
80	221.3	45.78
90	221.2	45.71
100	221.0	45.58

[a]Compared to uniform beam. [b]$(\omega_1)_{\text{uniform}} = 151.8$ rad/s.

Table 2 Numerical results for constrained optimization

Element no.	Simply supported[a] t_f, in.	Cantilever[b] t_f, in.
1	0.0694	0.0771
2	0.0253	0.0638
3	0.0118	0.0499
4	0.0093	0.0352
5	0.0091	0.0218
6	0.0091	0.0004
7	0.0093	0.0004
8	0.0118	0.0004
9	0.0253	0.0004
10	0.0694	0.0004

[a]See Fig. 2. [b]See Fig. 6

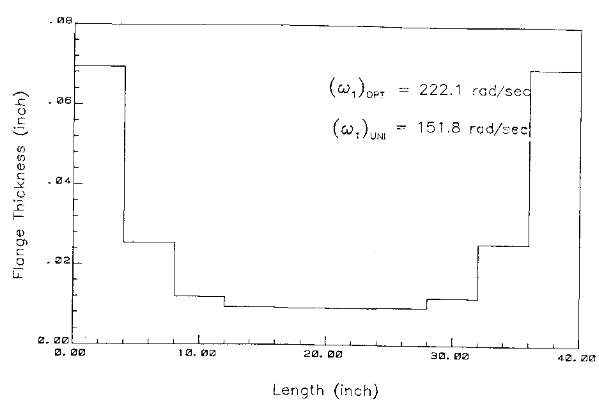

Fig. 2 Optimum flange thickness distribution of a simply supported beam: 10 elements.

However, because decoupled optimization is presumably inadequate, it follows that, at some point in the motion through design space, there will be a design for which $\omega_w^2 = \omega_v^2$. That design, although better than the initial uniform thickness design, is not optimum; and, if an even better design is desired, the movement in design space must satisfy the new active constraint of $\omega_w^2 = \omega_v^2$. This requires another optimality condition developed as follows. The new modified frequency function is

$$F = \int_L EI_z\left(\frac{d^2 v_r}{dx^2}\right)^2 dx - \lambda\left(\int_L \rho A\, dx - \bar{M}\right) - \Omega\left(\int_L \rho A v_r^2\, dx - 1\right) - \beta\left\{\int_L \left[EI_y\left(\frac{d^2 w_r}{dx^2}\right)^2 + 2EI_{zw}\frac{d^2 w_r}{dx^2}\frac{d^2\theta}{dx^2} + EC_{wr}\left(\frac{d^2\theta}{dx^2}\right)^2 + GJ\left(\frac{d\theta}{dx}\right)^2\right] dx - \int_L EI_z\left(\frac{d^2 v_r}{dx^2}\right)^2 dx\right\} - \alpha\left[\int_L \rho(Aw_r^2 + 2S_z w_r\theta + I_{pr}\theta^2)dx - 1\right] \tag{15}$$

which is simply the expression for ω_v^2 supplemented by the normality condition for v_r, the constant mass constraint, the constraint that $\omega_w^2 = \omega_v^2$, and the normality condition for w_r and θ. The variations δF_{v_r} leads to $(1+\beta)\omega_v^2 - \Omega = 0$. The variations δF_{w_r} and δF_θ lead to $\beta\omega_w^2 + \alpha = 0$. Finally, variation δF_{ϕ_j} leads to the new optimality criterion for each design variable:

$$\left(\frac{d^2 v_r}{dx^2}\right)^2 \frac{\partial(EI_z)}{\partial\phi_j} - \omega^2\rho v_r^2\frac{\partial A}{\partial\phi_j} - \lambda\rho\frac{\partial A}{\partial\phi_j} - \beta\left[\left(\frac{d^2 w_r}{dx^2}\right)^2\frac{\partial(EI_y)}{\partial\phi_j} + 2\frac{d^2 w_r}{dx^2}\frac{d^2\theta}{dx^2}\frac{\partial(EI_{zw})}{\partial\phi_j} + \left(\frac{d^2\theta}{dx^2}\right)^2\frac{\partial(EC_{wr})}{\partial\phi_j} + \left(\frac{d\theta}{dx}\right)^2\frac{\partial(GJ)}{\partial\phi_j} - \left(\frac{d^2 v_r}{dx^2}\right)^2\frac{\partial(EI_z)}{\partial\phi_j} - \omega^2\rho\left(w_r^2\frac{\partial A}{\partial\phi_j} + 2w_r\theta\frac{\partial S_z}{\partial\phi_j} + \theta^2\frac{\partial I_{pr}}{\partial\phi_j} - v_r^2\frac{\partial A}{\partial\phi_j}\right)\right] = 0 \tag{16}$$

The first terms of Eq. (16) are associated with optimizing ω_v^2 alone. The remaining terms, with the Lagrange multiplier β, appear because of the additional constraint that $\omega_v^2 = \omega_w^2 = \omega^2$.

For this coupled optimization problem, the unknowns have been augmented by the additional Lagrange multiplier β and an additional frequency of vibration ω, but the equations have been augmented by an additional normality equation and the constraint equation of equality of frequencies. The problem remains conceptually solvable, but the solution will be more difficult because of the second Lagrange multiplier.

III. Development of a Finite-Element Model

A channel cross section with constant specified web depth h and constant specified flange width b is considered. For numerical results to be presented, the beam is modeled as a collection of finite elements, and it is necessary to develop proper stiffness and mass matrices for each element.

If the thicknesses t_f and t_w have some specified variation within each finite element, such as a linear variation, then displacement-based finite-element stiffness and mass matrices can be developed from the energy definitions in the Appendix. However, in this paper, the optimization is based on finite elements with uniform thicknesses. Therefore, appropriate matrices have been formulated by taking available matrices based on shear center displacements w_s, θ, v_s and transforming to reference axis displacements w_r, θ, v_r as follows.

Matrices $[\bar{K}^e_{ws}]$ and $[\bar{M}^e_{ws}]$ denote 8×8 element stiffness and mass matrices developed with nodal degrees of freedom w_s, dw_s/dx, θ, $d\theta/dx$. At each node, the transformation from reference point r to shear center s is

$$\begin{Bmatrix} w_s \\ \dfrac{dw_s}{dx} \\ \theta \\ \dfrac{d\theta}{dx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -e & 0 \\ 0 & 1 & 0 & -e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w_r \\ \dfrac{dw_r}{dx} \\ \theta \\ \dfrac{d\theta}{dx} \end{Bmatrix} \tag{17}$$

where

$$e = (I_{z\omega}/I_y) = 3t_f b^2/(6t_f b + t_w h) \tag{18}$$

locates the shear center for each finite-element cross section (Fig. 1). In condensed notation, Eq. (17) is written as

$$\{\bar{w}_s\} = [\bar{T}]\{\bar{w}_r\} \tag{19}$$

where $\{\bar{w}_s\}$ and $\{\bar{w}_r\}$ denote 4×1 displacement vectors at a single node, and $[\bar{T}]$ is the 4×4 transformation array. The two nodal displacement vectors are combined to give 8×1 total element displacement vectors $\{w_s\}$ and $\{w_r\}$, which are related by a properly constructed 8×8 transformation $[T]$ as follows:

$$\{w_s\} = [T]\{w_r\} \tag{20}$$

Finally, the transformed stiffness and mass matrices are

$$[K^e_{wr}] = [T]^T[\bar{K}^e_{wr}][T] \tag{21}$$

$$[M^e_{wr}] = [T]^T[\bar{M}^e_{wr}][T] \tag{22}$$

The transformed elemental matrices of Eqs. (21) and (22) can now be merged in the usual manner to form the total structure matrices $[K_w]$ and $[M_w]$.

The uncoupled beam vibration in the y direction can be treated with the usual stiffness and mass matrices $[K_v]$ and $[M_v]$. Note that there will be only two degrees of freedom at each node, v_s and dv_s/dx.

IV. Finite-Element Formulation of the Primary Problem

A channel cross-section beam is considered to be composed of a specified number of finite elements with possibly differing values of web thickness t^e_w, flange thickness t^e_f, and length L^e. (The superscript e denotes element values.) A formulation with variable lengths L^e is given in Ref. 25, but, in this paper, the finite-element lengths are equal and specified. Therefore, the problem is to determine the set of wall thicknesses which will provide a maximum value for the fundamental frequency of vibration subject to the constraint of constant total volume (for uniform density material). In addition, there may be the so-called coupling constraint if the optimum design occurs with $\omega_w^2 = \omega_v^2$, as discussed earlier.

For the problem of optimizing the coupled bending-torsion frequency ω_w^2 without the constraint of $\omega_v^2=\omega_w^2$, the modified objective function, which is the finite-element form of Eq. (10), is given by

$$F(t_r^e,q_i)=K_{ij}(t_r^e)q_iq_j-\lambda\left[\sum_e A^e(t_r^e)L^e-\bar{V}\right]$$
$$-\Omega[M_{ij}(t_r^e)q_iq_j-1] \qquad (23)$$

where

K_{ij},M_{ij} = element in the ith row and the jth column of the total beam stiffness and mass matrices, respectively; associated with coupled bending-torsion vibration

t_r^e = rth design variable (t_w^e or t_f^e) in element e

There is the additional constraint that

$$(K_{ij}-\omega^2M_{ij})q_j=0 \qquad (24)$$

Note the use of the summation convention in Eqs. (23) and (24).

The first necessary condition for a differentiable maximum of F is $\partial F/\partial q_i=0$, from which it follows, after substitution from Eq. (24), that

$$\Omega=\omega^2 \qquad (25)$$

The next requirement is $\partial F/\partial t_r^e=0$, from which follows the optimality criteria given in Eqs. (28). In developing those equations, there will be terms of the form $[\partial K_{ij}/\partial t_r^e]q_iq_j$. Note, however, that the design variables t_r^e occur only in element e. Therefore, the derivatives involve only the appropriate stiffness and mass matrices for element e, and the only degrees of freedom that need be considered are those associated with element e. This means that the derivative terms can be written as $[\partial K_{ij}^e/\partial t_r^e]q_i^eq_j^e$, as shown in Eqs. (28).

The formulation can be summarized as follows. The unknowns are N_q values of q_i, N_t values of t_r^e, one value of ω^2, and one value of λ. The equations are as follows:

N_q equilibrium

$$(K_{ij}-\omega^2M_{ij})q_j=0, \qquad i=1,2,\ldots,N_q \qquad (26)$$

One normalization

$$M_{ij}q_iq_j-1=0 \qquad (27)$$

N_t optimality

$$\left(\frac{\partial A^e}{\partial t_r^e}L^e\right)^{-1}\left(\frac{\partial K_{ij}^e}{\partial t_r^e}-\omega^2\frac{\partial M_{ij}^e}{\partial t_r^e}\right)q_i^eq_j^e-\lambda=0$$
$$r=1,2,\ldots,N_t \qquad (28)$$

One constraint

$$\sum_e A^eL^e-\bar{V}=0 \qquad (29)$$

A simultaneous solution of Eqs. (26-29) will lead to possible optimum designs.

When speaking of N_q equations of equilibrium, as in Eq. (26) and subsequently, there are of course only N_q-1 independent equations. The remaining needed equation is the characteristic equation established from vanishing of the appropriate determinant.

Some of the design variables might take on specified values, such as a thickness equal to a lower limit value. If this occurs, simply give those variables the specified values wherever they occur and remove the optimality criteria associated with differentiation with respect to those variables.

The next case to investigate is when the optimum design occurs with $\omega_w^2=\omega_v^2$, and the modified objective function, which is the finite-element form of Eq. (15), is

$$F(t_r^e,q_{wi},q_{vi})=K_{vij}q_{vi}q_{vj}-\lambda\left(\sum_e A^e-\bar{V}\right)-\Omega(M_{vij}q_{vi}q_{vj}-1)$$
$$-\beta(K_{wij}q_{wi}q_{wj}-K_{vij}q_{vi}q_{vj})-\alpha(M_{wij}q_{wi}q_{wj}-1) \qquad (30)$$

It is now necessary to distinguish between coupled bending-torsion vibration, denoted by subscript w, and the uncoupled bending, denoted by subscript v. The derivatives with respect to q_{vi} lead to $(1+\beta)\omega_v^2-\Omega=0$, and the derivatives with respect to q_{wi} lead to $\beta\omega_w^2+\alpha=0$. The derivatives with respect to design variables t_r^e lead to the optimality criteria shown in Eqs. (35).

This coupled optimization problem is summarized as follows. The unknowns are N_w values of q_{wi}, N_v values of q_{vi}, N_t values of t_r^e, two values of ω^2, one value of λ, and one value of β. The equations are as follows:

N_w equilibrium

$$(K_{wij}-\omega^2M_{wij})q_{wj}=0, \qquad i=1,2,\ldots,N_w \qquad (31)$$

One normalization

$$M_{wij}q_{wi}q_{wj}-1=0 \qquad (32)$$

N_v equilibrium

$$(K_{vij}-\omega^2M_{vij})q_{vj}=0, \qquad i=1,2,\ldots,N_v \qquad (33)$$

One normalization

$$M_{vij}q_{vi}q_{vj}-1=0 \qquad (34)$$

N_t optimality

$$\left(\frac{\partial K_{vij}^e}{\partial t_r^e}-\omega^2\frac{\partial M_{vij}^e}{\partial t_r^e}\right)q_{vi}^eq_{vj}^e-\lambda\frac{\partial A^e}{\partial t_r^e}L^e$$
$$-\beta\left[\left(\frac{\partial K_{wij}^e}{\partial t_r^e}-\omega^2\frac{\partial M_{wij}^e}{\partial t_r^e}\right)q_{wi}^eq_{wj}^e\right.$$
$$\left.-\left(\frac{\partial K_{vij}^e}{\partial t_r^e}-\omega^2\frac{\partial M_{vij}^e}{\partial t_r^e}\right)q_{vi}^eq_{vj}^e\right]=0, \qquad r=1,2,\ldots,N_t \qquad (35)$$

One constraint

$$\sum_e A^eL^e-\bar{V}=0 \qquad (36)$$

One constraint

$$K_{wij}q_{wi}q_{wj}-K_{vij}q_{vi}q_{vj}=0 \qquad (37)$$

V. Recursion Relationship for the Primal Problem

For the primal problem with uncoupled optimization, the optimization process begins with some known distribution of design variables that satisfy the geometric constraints of Eqs.

(29). For this initial design, Eqs. (26) and (27) are solved for the eigenvalue ω and the associated normalized eigenvector q_i. Then it is possible to substitute into the optimality conditions of Eqs. (28). Only on rare occasions will these equations provide immediate solutions for the Lagrange multiplier λ, and so what is required is a procedure for moving through design variable space in such a manner as eventually to locate a design that permits satisfaction of the optimality criteria. This will be done with an iteration scheme developed as follows.

First introduce the definitions

$$U_r^e=\frac{\partial K_{ij}^e}{\partial t_r^e}\,q_i^e q_j^e,\qquad T_r^e=\frac{\partial M_{ij}^e}{\partial t_r^e}\,q_i^e q_j^e,\qquad A_r^e=\frac{\partial A^e}{\partial t_r^e} \tag{38}$$

Now the optimality criteria, Eqs. (28), can be written as

$$U_r^e-\omega^2 T_r^e-\lambda A_r^e L^e=0 \tag{39}$$

Next define

$$Z_r^e=U_r^e-\omega^2 T_r^e \tag{40}$$

so that the optimality criteria can be written as

$$Z_r^e-\lambda A_r^e L^e=0,\qquad r=1,2,\ldots,N_r \tag{41}$$

At the optimum design there will be a single value for λ which satisfies all N_r equations of Eqs. (41). However, for a nonoptimum design, there is no single value of λ; and what will prove useful is some type of "best" value for λ, say $\bar{\lambda}$, which approximately satisfies Eqs. (41) according to some criterion of goodness.

From Eqs. (41), define

$$\lambda_r=Z_r^e/A_r^e L^e \tag{42}$$

Evidently, λ_r is the estimate for Lagrange multiplier λ based on the rth equation. Then Eqs. (41) can be written as

$$(\lambda_r-\lambda)A_r^e L^e=0\qquad\text{or}\qquad \lambda_r-\lambda=0 \tag{43}$$

If the best value of λ is determined by the method of weighted residuals with equal weights for each equation, the result is

$$\bar{\lambda}=(1/N_r)\Sigma_r\lambda_r \tag{44}$$

so that $\bar{\lambda}$ is simply the arithmetic average of the λ_r.

The next step is to assume that the $(\nu+1)$th iteration values can be expressed in terms of the νth iteration as follows:

$$\begin{Bmatrix} U_r^e \\ T_r^e \end{Bmatrix}^{\nu+1}=\left[\frac{(t_r^e)^\nu}{(t_r^e)^{\nu+1}}\right]^\eta \begin{Bmatrix} U_r^e \\ T_r^e \end{Bmatrix}^{\nu} \tag{45}$$

where η is a positive exponent. No attempt is made to derive these relationships. For some optimization problems in which the optimality criteria can be expressed in terms of potential and kinetic energies, it is possible to make some plausibility arguments relating $(\nu+1)$ and ν energies. These arguments are simply carried without change to this problem for which the optimality criteria cannot be expressed in terms of energy, leading to Eqs. (45). The only proof of validity is utilitarian: Do the assumed relationships lead to procedures that do indeed achieve optimum design?

Substitution of Eqs. (45) into Eq. (40) gives

$$(Z_r^e)^{\nu+1}=\left[\frac{(t_r^e)^\nu}{(t_r^e)^{\nu+1}}\right]^\eta (Z_r^e)^\nu \tag{46}$$

Note that $(\omega^2)^\nu$ is used in the definiton of $(Z_r^e)^{\nu+1}$. Substitute Eq. (46) into Eq. (42), and get the following approximation for $\lambda_r^{\nu+1}$:

$$\lambda_r^{\nu+1}=\frac{(Z_r^e)^{\nu+1}}{(A_r^e L^e)^\nu}=\left[\frac{(t_r^e)^\nu}{(t_r^e)^{\nu+1}}\right]^\eta \lambda_r^\nu \tag{47}$$

Now, the new design variables are selected so that the $\lambda_r^{\nu+1}$ as defined are equal to each other for all values of r. This movement toward equality of λ_r is expected to be a movement toward the optimum design. The equal value is chosen to be $\bar{\lambda}^\nu$ so that

$$\lambda_r^{\nu+1}=\left[\frac{(t_r^e)^\nu}{(t_r^e)^{\nu+1}}\right]^\eta \lambda_r^\nu=\bar{\lambda}^\nu \tag{48}$$

Therefore, the $(\nu+1)$ values can be written in terms of the ν values as follows:

$$(t_r^e)^{\nu+1}=af_r^\nu(t_r^e)^\nu\qquad\text{with}\qquad f_r^\nu=(\lambda_r^\nu/\bar{\lambda}^\nu)^n \tag{49}$$

where $n=1/\eta$ is positive.

Equation (49) includes a scalar multiplier that is used to force the $(\nu+1)$ design variables to satisfy the volume constraint. Because there might be active geometric constraints of the type of Eq. (9) acting on some of the design variables, the volume constraint can be written as

$$\sum_{e=1}^{N_e-N_{cr}} A^e[(t_r^e)^{\nu+1}(L^e)]=\bar{V}-V_c \tag{50}$$

Substitution of Eq. (49) into Eq. (50) gives

$$a=(\bar{V}-V_c)\Big/\sum_{e=1}^{N_e-N_{cr}} A^e[f_r^\nu(t_r^e)^\nu](L^e) \tag{51}$$

Note that, when developing $A^e[(t_r^e)^{\nu+1}]$, it is recognized that the cross-section area is a linear function of design variable t_r^e for the channel section with constant h and b [see Eq. (A3)].

Equation (49) is useful only when the quantities f_r^ν are defined, which requires $(\lambda_r^\nu/\bar{\lambda}^\nu)$ to be defined and positive. If these requirements are not satisfied, then proceed as follows. Write Eq. (39) in the following forms:

If $\bar{\lambda}^\nu>0$, $\lambda_r^\nu<0$, then

$$(U_r^e)^{\nu+1}=(\omega^2)^\nu(T_r^e)^\nu+\bar{\lambda}^\nu(A_r^e)^\nu L^e \tag{52a}$$

$$(U_r^e)^{\nu+1}=(U_r^e)^\nu+(\bar{\lambda}^\nu-\lambda_r^\nu)(A_r^e)^\nu L^e \tag{52b}$$

If $\bar{\lambda}^\nu<0$, $\lambda_r^\nu>0$, then

$$(\omega^2)^\nu(T_r^e)^{\nu+1}=(U_r^e)^\nu+(-\bar{\lambda})^\nu(A_r^e)^\nu L^e \tag{53a}$$

$$(\omega^2)^\nu(T_r^e)^{\nu+1}=(\omega^2)^\nu(T_r^e)^\nu+(\lambda_r^\nu-\bar{\lambda}^\nu)(A_r^e)^\nu L^e \tag{53b}$$

From positive definiteness of energies, it follows that U_r^e, T_r^e are all positive definite. Also, $A_r^e>0$ always. Clearly Eqs. (52) and (53) have been written in such a way as to guarantee positive quantities on each side of each equation. In each case, the intention is for the $(\nu+1)$ design to be such that the left-hand side will be increased to the ν value on the right-hand side. Substitution from Eqs. (45) and introduction of the scalar a gives the following results:

If $\bar{\lambda}^\nu>0$, $\lambda_r^\nu<0$, then

$$(t_r^e)^{\nu+1}=a\left[\frac{(U_r^e)^\nu}{(U_r^e)^\nu+(\bar{\lambda}^\nu-\lambda_r^\nu)(A_r^e)^\nu L^e}\right]^n (t_r^e)^\nu \tag{54}$$

If $\bar{\lambda}^\nu < 0$, $\lambda_r^\nu > 0$, then

$$(t_r^e)^{\nu+1} = a\left[\frac{(\omega^2)^\nu (T_r^e)^\nu}{(\omega^2)^\nu (T_r^e)^\nu + (\lambda_r^\nu - \bar{\lambda}^\nu)(A_r^e)^\nu L^e}\right]^n (t_r^e)^\nu \qquad (55)$$

The scalar a is again found from Eqs. (51) with proper redefinition for the quantities f_r^ν.

In summary, the recursion relations are as follows. Use Eqs. (49) if valid, because those equations account for simultaneous changes in both U_r^e and T_r^e. These equations should certainly be valid when the design becomes sufficiently close to an optimum design. If, in the early stages of the iteration process, Eqs. (49) are not valid for some design variables, then use Eqs. (54) and (55) as appropriate to modify those particular variables.

VI. Numerical Results

The channel cross section of the following dimensions has been considered (Fig 1): $h = 0.5$ in., $b = 0.975$ in., $t_w = 0.025$ in., $E = 10\times 10^6$ psi, $G = 3.8\times 10^6$ psi, $\rho = 0.243\times 10^{-3}$ lb-s^2/in.[4] The beam length is 40 in., and that length has been divided into equal-length finite elements. The only design variable for each finite element is the flange thickness t_f. With the area given by Eq. (A3), the volume constraint of Eq. (50) reduces to

$$\sum_{e=1}^{N_e - N_{cr}} (c_1 + c_2 t_f^e) = \frac{(\bar{V} - V_c)}{L^e} \qquad (56)$$

where $c_1 = 0.0125$, $c_2 = 1.95$, and $\bar{V} = 1.0$ in.^{3}. Both simply supported and cantilever boundary conditions have been studied. For simple support, the minimum thickness is $t_f = 0.0005$ in., whereas for the cantilever beam, minimum thickness is $t_f = 0.0004$.

For the results to be presented, the optimization process started with a uniform wall thickness, which means $t_f = t_w = 0.025$ in. The recursion relations are Eqs. (49), (54), or (55), as appropriate, with $t_r^e = t_f^e$ and $A_r^e = c_2$. The scaling factor is given by Eq. (51).

The optimum design is identified by satisfaction of the optimality criterion in the form of Eq. (43), which requires a constant value for all λ_r. With one design variable per finite element, it follows that there will be one λ_r per element, and the uniformity of those λ_r can be evaluated by the requirement

$$\left|\left(\frac{\lambda_r|_{\max}}{\lambda_r|_{\min}}\right) - 1\right| < \epsilon \qquad (57)$$

where $\epsilon = 0.001$ is a measure of acceptable error.

Convergence to the optimum frequency was smooth and monotonic. The rate of convergence was a function of the initial choice of the exponent n, which appears in the recursion relations, and also a function of how n was modified. In this work, n was initially set to a value of 1.0. During the iteration process, if at any stage $(\omega)^{\nu+1}$ is less than $(\omega)^\nu$, then the value of n is reduced by 75%.

The optimal flange thicknesses of the simply supported beam, corresponding to a few of the discretized cases, are shown in Figs. 2–5. Figure 6 shows the optimum flange thickness distribution of the cantilever beam. The results are summarized in Tables 1 and 2. For the simply supported beam, it is observed from Table 1 that the 10-element discretized beam yields the largest increase in the first frequency ω_1 when compared to the corresponding value for a beam with uniform flange thickness. This could be because a 10-element discretized beam is not an exact representation of the continuous beam, and since it has less degrees of freedom than beams with larger discretization, it yields a higher frequency. However, the fact remains that, for a given mass, a beam with 10 stepwise thickness distributions (Fig. 2) gives the highest increase in ω_1. In order to check if further refinement of this stepped beam leads to higher frequency, the beam was discretized into 100 elements. However, no further increase in ω_1 was observed. In case of the cantilever beam, an increase of 178.9% in the first frequency ω_1 is realized using 10 finite elements, when compared to the corresponding value for a beam with uniform flange thickness.

Some important observations regarding the optimum design variable distributions are made at this point. In the case of the solid, simply supported beam undergoing flexural vibration, the optimum area distribution corresponding to

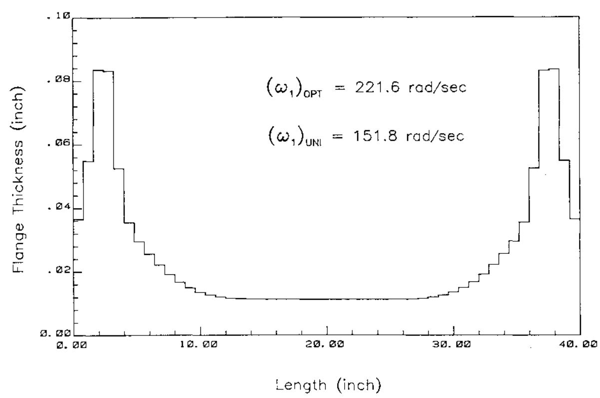

Fig. 4 Optimum flange thickness distribution of a simply supported beam: 50 elements.

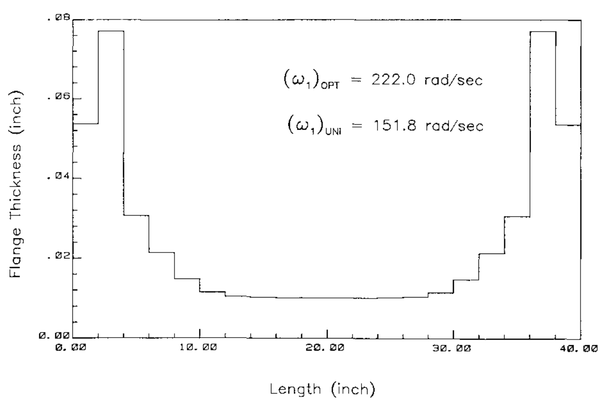

Fig. 3 Optimum flange thickness distribution of a simply supported beam: 20 elements.

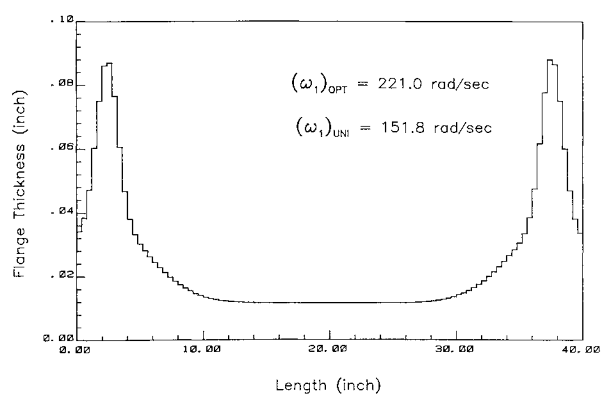

Fig. 5 Optimum flange thickness distribution of a simply supported beam: 100 elements.

the maximum fundamental frequency follows the pattern of the corresponding mode shape. In other words, the optimum distribution assumes a maximum at the center with minimum at the two ends (Fig. 7). The flange thickness distribution corresponding to the optimum fundamental frequency of the simply supported channel section, however, assumes a minimum at the center, with maximum toward the two ends (Figs. 2-5). The difference is attributed to the following reason. A beam with a thin-walled open section such as a channel is very weak in resistance to torsion. So, the fundamental mode of coupled vibration is a predominantly torsion-dominated mode. Since the twisting moment distribution of a simply supported channel beam has its maximum at the two ends and a minimum at the center, the optimum distribution tends to follow this pattern. Also, for solid sections, beams with second area moments of inertia proportional to the square of the cross-sectional area have been considered, whereas, in the case of the beam with channel cross section, the design variable yields a linear relation of the type

$$I(x) = \alpha_0 + \alpha_1 A(x) \tag{58}$$

which may also contribute toward changing the nature of the optimum distribution.

For cantilever channel beams, the optimum flange thickness distribution is similar in nature to that of the solid cantilever undergoing only flexural motion. The reason for this is that, although the first coupled mode of vibration is still a torsion-dominated mode, the twisting moment distribution in the case of a cantilever beam has its maximum at the root and minimum at the free end. Although the optimum distribution tends to follow the torsion-dominated first natural mode, it is similar in pattern to the optimum distribution of a solid cantilever under bending only (Fig. 6 and 8).

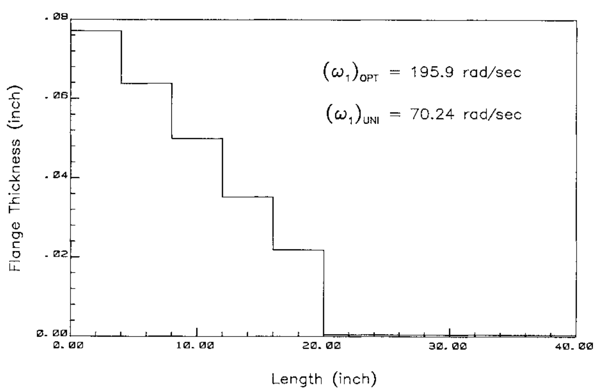

Fig. 6 Optimum flange thickness distribution of a cantilever beam: 10 elements

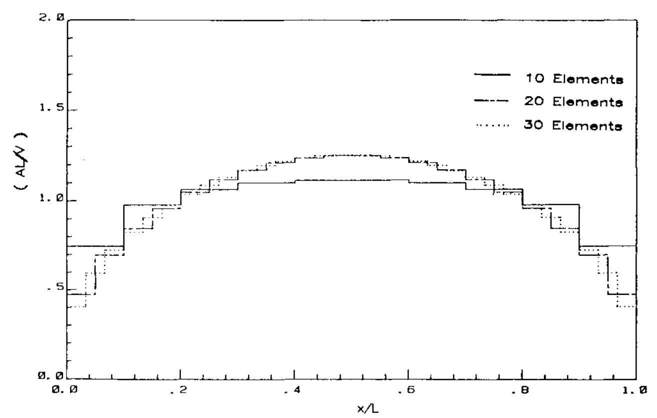

Fig. 7 Optimum area distribution of a simply supported vibrating beam in ω_1^2 maximization case (solid section $I = cA^2$).

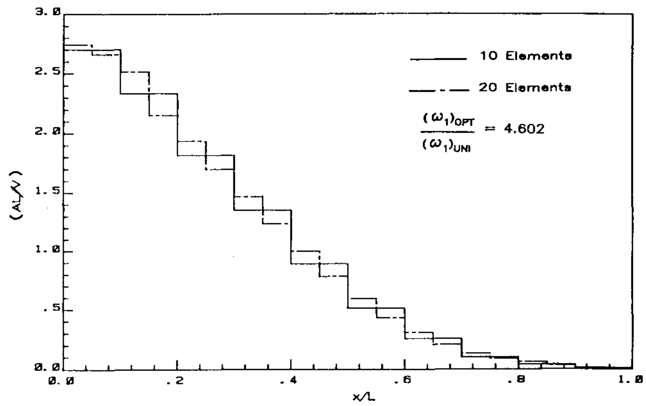

Fig. 8 Optimum area distribution of a vibrating cantilever in ω_1^2 maximization case (solid section $I = cA^2$).

VII. Conclusions

In this paper, an optimality criterion approach has been developed to maximize the fundamental frequency of a thin-walled beam with coupled bending and torsional modes. The results show that the optimum designs, in some cases, are very different from the designs obtained for beams with uncoupled vibrations. This suggests further studies in this field, including the dual problem of minimizing the weight for frequency restraints: beams with closed cross sections and multiple frequency constraints. Future studies may also include the possiblities of using the length of the element, in the finite-element discretization of the continuous system, as an additional design variable. In practical applications where the coupling of bending and torsional modes cannot be avoided, such as in rotorcraft technology, any analysis that ignores the effect of coupling may lead to erroneous results.

Appendix: Free Vibrations of Channel Sections with Nonuniform Wall Thickness

A beam of channel cross section with dimensions and coordinate system as shown in Fig. 1 is considered. The web depth h and the flange width b are constant, but the thicknesses t_w and t_f are nonuniform along the length of the beam. At each cross section, the shear center s and the centroid c are located by their y cooordinates:

$$e = 3b^2 [6b + (ht_w/t_f)]^{-1} \tag{A1}$$

$$\bar{y} = b^2 [2b + (ht_w/t_f)]^{-1} \tag{A2}$$

If the ratio t_w/t_f is constant, then the loci of shear centers and centroids will be straight lines, and shear center displacements will provide elastic decoupling of rotation and z-direction displacement just as for a beam that is uniform along the length. However, for more general axial variations of thickness, the shear centers will lie on a curved line that is not so suitable for the beam reference axis. Therefore, the reference axis should be chosen so as to be straight for any variation of thickness; and, for the channel considered in this paper, an appropriate reference axis passes through the web center at each cross section. The purpose of this Appendix is to present the necessary equations that describe free vibration in terms of displacements at the web center.

The fundamental assumptions are the usual two assumptions for thin-walled beams. First, each cross section is assumed to twist without distortion. Second, there is no shear deformation in the middle surface of the thin walls of the beam.

At each cross section, it is possible to have variation in the elastic moduli E and G, the mass density ρ, and the wall thicknesses t_w and t_f. This general case is presented in Ref. 25. This paper is limited to the cases in which all of these quantities are constant within a given cross section; and, for these beams, the following geometric properties naturally appear in the development of the equations:

$$A = 2t_f b + t_w h \tag{A3}$$

$$S_z = t_f b^2 \tag{A4}$$

$$I_y = (t_f b h^2/2) + (t_w h^3/12) \tag{A5}$$

$$\bar{I}_z = 2t_f b^3/3 \tag{A6}$$

$$I_z = \bar{I}_z - S_z^2/A \tag{A7}$$

$$I_{pr} = I_y + \bar{I}_z \tag{A8}$$

$$J = (t_w^3 h/3) + (2t_f^3 b/3) \tag{A9}$$

$$I_{zw} = -t_f h^2 b^2/4 \tag{A10}$$

$$C_{wr} = t_f h^2 b^3/6 \tag{A11}$$

The moment-displacement relations are

$$M_y = EI_y \frac{\partial^2 w_r}{\partial x^2} + EI_{z\omega} \frac{\partial^2 \theta}{\partial x^2} \tag{A12}$$

$$M_{wr} = EI_{z\omega} \frac{\partial^2 w_r}{\partial x^2} + EC_{wr} \frac{\partial^2 \theta}{\partial x^2} \tag{A13}$$

$$M_{xsv} = GJ \frac{\partial \theta}{\partial x} \tag{A14}$$

$$M_z = EI_z \frac{\partial^2 v_r}{\partial x^2} \tag{A15}$$

The equations of dynamic equilibrium, as derived from a differential beam element, can be written in the form

$$\frac{\partial^2 M_y}{\partial x^2} + \rho A \frac{\partial^2 w_r}{\partial t^2} + \rho S_z \frac{\partial^2 \theta}{\partial t^2} = 0 \tag{A16}$$

$$\frac{\partial^2 M_{wr}}{\partial x^2} - \frac{\partial M_{xsv}}{\partial x} + \rho I_{pr} \frac{\partial^2 \theta}{\partial t^2} + \rho S_z \frac{\partial^2 w_r}{\partial t^2} = 0 \tag{A17}$$

$$\frac{\partial^2 M_z}{\partial x^2} + \rho A \frac{\partial^2 v_r}{\partial t^2} = 0 \tag{A18}$$

To complete the formulation, certain conditions must be satisfied at each point on the boundary. These boundary conditions consist of specified values of six quantities selected appropriately from the following list:

$$\text{either} \quad \frac{\partial M_y}{\partial x} \quad \text{or} \quad w_r \tag{A19}$$

$$\text{either} \quad M_y \quad \text{or} \quad \frac{\partial w_r}{\partial x} \tag{A20}$$

$$\text{either} \quad \frac{\partial M_{wr}}{\partial x} \quad \text{or} \quad \theta \tag{A21}$$

$$\text{either} \quad M_{wr} \quad \text{or} \quad \frac{\partial \theta}{\partial x} \tag{A22}$$

$$\text{either} \quad \frac{\partial M_z}{\partial x} \quad \text{or} \quad v_r \tag{A23}$$

$$\text{either} \quad M_z \quad \text{or} \quad \frac{\partial v_r}{\partial x} \tag{A24}$$

The strain energy in the beam is given by

$$U = \frac{1}{2} \int_L \left[EI_y \left(\frac{\partial^2 w_r}{\partial x^2} \right)^2 + 2EI_{z\omega} \frac{\partial^2 w_r}{\partial x^2} \frac{\partial^2 \theta}{\partial x^2} + EC_{wr} \left(\frac{\partial^2 \theta}{\partial x^2} \right)^2 + GJ \left(\frac{\partial \theta}{\partial x} \right)^2 + EI_z \left(\frac{\partial^2 v_r}{\partial x^2} \right)^2 \right] dx \tag{A25}$$

where the first term is due to the bending in the z direction (see Fig. 1), the second term is due to the coupling between bending and torsion, the third term is the contribution of the warping, the fourth term is due to torsion, and the fifth term is due to bending in the y direction.

The kinetic energy is given by

$$T = \frac{1}{2} \int_L \left[\rho A \left(\frac{\partial w_r}{\partial t} \right)^2 + 2\rho S_z \frac{\partial w_r}{\partial t} \frac{\partial \theta}{\partial t} + \rho I_{pr} \left(\frac{\partial \theta}{\partial t} \right)^2 + \rho A \left(\frac{\partial v_r}{\partial t} \right)^2 \right] dx \tag{A26}$$

where the first term is due to bending in the z direction, the second term is the contribution of the coupling present, the third term is due to torsion, and the fourth term is due to bending in the y direction.

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